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ON THE STABILITY OF MOTION OF A GYROSCOPE IN GIMBALS AND THE EVALUATION OF DEFLECTIONS*

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The problem of stability of gyroscope motion is considered in the formulation in /1/. Liapunov function is constructed in the form of a nonlinear bundle of integrals. The obtained sufficient conditions of regular precession stability of a gyroscope coincide with those obtained in /1/ using the Routh theorem. Deviations from stable motion of the gyroscope are evaluated on the basis of the method of /2/ with allowance for the findings of /3/.

Let the system of differential equations of perturbed motion allow the Liapunov function to be constructed in the quadratic form

$$V = a_1x_1^2 + a_1x_2^2 + \ldots + a_nx_n^2$$

where the coefficients a_i are positive, continuous, and differentiable functions of some parameter λ. The relation between initial and current perturbations is assumed to be of the form /2/

$$H_0 = hH; \quad h = \max\ f(\lambda) \tag{1}$$

$$f(\lambda) = \frac{\min\{a_1(\lambda), a_2(\lambda), \ldots a_n(\lambda)\}}{\max\{a_1(\lambda), a_2(\lambda), \ldots, a_n(\lambda)\}} \tag{2}$$

The domain of determination of function $f(\lambda)$ can be represented as a set of intervals on the λ axis, where curves $a_i(\lambda)$ are positive and do not intersect each other. Thus the selection of parameter λ enables us to determine the maximum radius of the sphere of initial perturbations for which the moving point will not go outside the limits of the sphere of a given radius.

Let a gyroscope suspended in gimbals be in a field of forces with force function $u(\theta)$. We shall carry out the analysis in the fixed system of coordinates $OXYZ$ whose OZ axis lies on the axis of gimbals outer ring, and the moving system of coordinates $Oxyz$ whose axes are rigidly attached to the inner gimbals ring and are the principal axes of inertia of the gyroscope and the inner ring. The respective positions of axes of coordinates $OXYZ$ and $Oxyz$ can be defined by Euler's angles θ, ψ, φ, where ψ is the angle of turn of the external gimbals ring, θ is the angle of turn of the casing in the ring, and φ is the angle of turn of the gyroscope relative to the system $OXYZ$ (the angle of proper rotation).

The equations of motion admit the first integrals

$$\begin{aligned}
&(A + A_1)\theta'^2 + (A + B_1 - C_1)\psi'^2\sin^2\theta + C_1\psi'^2\cos^2\theta + C(\varphi' + \psi'\cos\theta)^2 + J\psi'^2 - 2U = \text{const} \\
&(A + B_1 - C_1)\psi'\sin^2\theta + C(\varphi' + \psi'\cos\theta)\cos\theta + C_1\psi' + J\psi' = \text{const} \\
&\varphi' + \psi'\cos\theta = \text{const}
\end{aligned} \tag{3}$$

The first of these is the energy integral and the remaining two correspond to ignorable coordinates ψ and φ. In these equations $A, B = A, C$ and A_1, B_1, C_1 are the principal moment of inertia of the gyroscope and of the inner ring, respectively, and J is the moment of inertia of the outer ring relative to the OZ axis.

When the constants θ_0, ψ_0', r_0 satisfy the equation

$$(A + B_1 - C_1)\psi_0'^2\sin\theta_0\cos\theta_0 - Cr_0\psi_0'\sin\theta_0 + (\partial u/\partial\theta)_{\theta=\theta_0} = 0 \tag{4}$$

the equations of motion admit the particular solution

$$\theta = \theta_0, \quad \theta' = 0, \quad \psi' = \psi_0', \quad r = r_0$$

which under condition $\theta_0 = 0$ represents uniform rotation and, if $\theta_0 \neq 0, \pi$ is the regular precession of the gyroscope.

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Let us consider the perturbed motion

$$\theta = \theta_0 + x_1, \quad \theta' = x_2, \quad \psi' = \psi_0' + x_3, \quad r = r_0 + x_4$$

Using the integrals of perturbed motion with terms of up to the second order /4/, we construct a Liapunov function of the form

$$V = \sum_{r=1}^{3} \alpha_r V_r + \sum_{m,l=1}^{3} \beta_{ml} V_m V_l \tag{5}$$

With condition (4) satisfied and parameters

$$\alpha_1 = 1, \quad \alpha_2 = -2\psi_0', \quad \alpha_3 = 2C(\psi_0' \cos\theta_0 - r_0)$$

the expansion of function (5) in series of powers of x_i begins with second order terms to which the analysis can be restricted on the assumption of smallness of generated perturbations.

We introduce the notation

$$\gamma_0 = -(A + B_1 - C_1)\psi_0'^2 (\cos^2\theta_0 - \sin^2\theta_0) + Cr_0\psi_0' \cos\theta_0 - (\partial^2 u/\partial x_1^2)_0$$
$$d_1 = [2(A + B_1 - C_1)\psi_0' \cos\theta_0 - Cr_0] \sin\theta_0$$
$$d_2 = J + C_1 + (A + B_1 - C_1)\sin^2\theta_0$$

We set $\beta_{11} = \beta_{12} = 0, \beta_{22} = \lambda$, and select the remaining parameters β_{ml} so that equalities

$$2\lambda C \cos\theta_0 + 2\psi_0'\beta_{13} + \beta_{23} = 0$$
$$C + 4(C\cos\theta_0)^2\lambda + \beta_{33} + 4Cr_0\beta_{13} + 4C\cos\theta_0\beta_{23} = a$$

where a is some so far undetermined quantity, are satisfied.

The coefficients at quadratic terms of function (5) are of the form

$$\gamma_{11} = \gamma_0 + d_1^2\lambda, \quad \gamma_{22} = A + A_1, \quad \gamma_{33} = d_2 + d_2^2\lambda, \quad \gamma_{44} = a$$
$$\gamma_{13} = d_1 d_2 \lambda, \quad \gamma_{14} = \psi_0' \sin\theta_0, \quad \gamma_{34} = 0 \tag{6}$$

We select the quantity a so that the equality $\gamma_{33} = \gamma_{44}$ be satisfied. The quadratic form with coefficients (6) is positive definite when conditions

$$\gamma_0 d_2 + d_1^2 > 0, \quad \lambda(\gamma_0 d_2 + d_1^2) d_2 + \gamma_0 d_2 - \gamma_{14}^2 > 0 \tag{7}$$

are satisfied.

In the case of uniform rotation of the gyroscope the sufficient condition of that motion (7) is of the form

$$\gamma_0 > 0, \quad \lambda > -1/d_2 \tag{8}$$

and is the same as the condition obtained in /1,4,5/. When the gyroscope precession is regular, conditions (7) coincide with the necessary and sufficient stability conditions obtained in /1/ using the Routh theorem.

Let us estimate the deviations of the gyroscope perturbed motion. To be able to compare deviations with respect to various coordinates it is necessary to introduce dimensionless parameters

$$\theta_1' = s\theta', \quad \psi_1' = s\psi', \quad r_1 = sr, \; t_1 = t/s$$
$$s = \max\{|\theta_0'|^{-1}, |\psi_0'|^{-1}, |r_0|^{-1}\} \tag{9}$$

The form of equations of motion and integrals (3) is not affected by the substitution of function $u_1 = s^2 u$, for the force function u. Part of coefficients (6) expressed in dimensionless parameters acquire new values, but for simplicity of notation we retain their previous symbols.

In the case of uniform rotation of the gyroscope coefficients (6) can be written as

$$\gamma_{11} = \gamma_0, \; \gamma_{22} = \gamma_{33} = \gamma_{44} = A + A_1, \; \gamma_{13} = \gamma_{14} = \gamma_{34} = 0$$
$$\lambda = d_2^{-2}(A + A_1 - d_2)$$

The lowest and the highest values of these yield an estimate of deviation of the gyroscope motion from its steady motion in the form (1).

When the gyroscope precession is uniform, we reduce the quadratic form with coefficients (6), expressed in dimensionless parameters (9), to a diagonal form with coefficients

$$a_{1,3} = {}^1/_2\{\gamma_{11} + \gamma_{33} \mp [(\gamma_{11} - \gamma_{33})^2 + 4(\gamma_{13}^2 + \gamma_{14}^2)]^{1/2}\}$$
$$a_2 = A + A_1, \; a_4 = \gamma_{33}$$

Since the inequalities $a_1(\lambda) < a_4(\lambda) < a_3(\lambda)$ are satisfied, it is possible to write function (2) as follows:

$$\frac{1}{2}(\gamma_0 + d_2) < A + A_1 < a_3(\lambda_0), \quad \lambda_0 = \frac{\gamma_{14}^2 - \gamma_0 d_2}{d_2(\gamma_0 d_2 + d_1^2)} \tag{13}$$

in the form

$$f(\lambda) = a_1(\lambda)/a_3(\lambda),\ \lambda > \lambda_0 \tag{11}$$

under condition

$$A + A_1 < {}^1/_2\,(\gamma_0 + d_2) \tag{12}$$

in the form

$$f(\lambda) = a_1(\lambda)/a_3(\lambda),\quad \lambda_0 < \lambda < \lambda_1;\quad f(\lambda) = A + A_1/a_3(\lambda),\ \lambda > \lambda_1$$

where λ_1 is the solution of equation $a_1(\lambda) = A + A_1$), and under condition

$$A + A_1 > a_3(\lambda_0) \tag{13}$$

in the form

$$f(\lambda) = a_1(\lambda)/A + A_1,\ \lambda_0 < \lambda < \lambda_2;\ f(\lambda) = a_1(\lambda)/a_3(\lambda),\ \lambda > \lambda_2$$

where λ_2 is the solution of equation $a_3(\lambda) = A + A_1$).

Let us evaluate the extremum of function (11). Equation $f'(\lambda) = 0$ can be reduced to the form

$$-\lambda\,(d_1^2 + d_2^2)(\gamma_0 d_2 + d_1^2)\,d_2 + (\gamma_0 - d_2)(\gamma_0 d_2 - d_1^2)\,d_2 + 2(d_1^2 + d_2^2)\,\gamma_{14}^2 = 0$$

whose solution λ_m is the value for which function (11) reaches its maximum value.

It is now possible to obtain the value of coefficient h which determines the relation between the radii of initial and current perturbation spheres using formula (1) and conditions (10), (12), and (13), respectively,

$$h = f(\lambda_m) \tag{14}$$

$$h = a_1(\lambda_m)/a_3(\lambda_m),\ \lambda_0 < \lambda_m < \lambda_1;\ h = (A + A_1)/a_3(\lambda_1),\ \lambda_m > \lambda_1 \tag{15}$$

$$h = a_1(\lambda_2)/A + A_1,\ \lambda_0 < \lambda_m < \lambda_2;\ h = a_1(\lambda_m)/a_3(\lambda_m),\ \lambda_m > \lambda_2 \tag{16}$$

It is, thus, possible to obtain in each of the considered cases an estimate of deviations of perturbed motion of a gyroscope in gimbals from the stable one.

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